Nonlocal Equation of State
in
Anisotropic Static Fluid Spheres in General Relativity

H. Hernández
Laboratorio de Física Teórica, Departamento de Física,
Facultad de Ciencias, Universidad de Los Andes,
Mérida 5101, Venezuela
and
Centro Nacional de Cálculo Científico
Universidad de Los Andes (CeCalCULA),
Corporación Parque Tecnológico de Mérida,
Mérida 5101, Venezuela

L.A. Núñez
Centro de Astrofísica Teórica, Facultad de Ciencias,
Universidad de Los Andes, Mérida 5101, Venezuela
and
Centro Nacional de Cálculo Científico
Universidad de Los Andes (CeCalCULA),
Corporación Parque Tecnológico de Mérida,
Mérida 5101, Venezuela

March 2002

Abstract

We show that it is possible to obtain credible static anisotropic spherically symmetric matter configurations starting from known density profiles and satisfying a nonlocal equation of state. These particular types of equation of state describe, at a given point, the components of the corresponding energy-momentum tensor not only as a function at that point, but as a functional throughout the enclosed configuration. To establish the physical plausibility of the proposed family of solutions satisfying nonlocal equation of state, we study the constraints imposed by the junction and energy conditions on these bounded matter distributions.

We also show that it is possible to obtain physically plausible static anisotropic spherically symmetric matter configurations, having nonlocal equations of state, concerning the particular cases where the radial pressure vanishes and, other where the tangential pressures vanishes. The later very particular type of relativistic sphere with vanishing tangential stresses is inspired by some of the models proposed to describe extremely magnetized neutron stars (magnetars) during the transverse quantum collapse.

1 Introduction

Classical continuum theories are based on the assumption that the state of a body is determined entirely by the behavior of an arbitrary infinitesimal neighborhood centered at any of its material points. Furthermore, there is also a premise that any small piece of the material can serve as a representative of the entire body in its behavior, hence the governing balance laws are assumed to be valid for every part of the body, no matter how small. Clearly, the influence of the neighborhood on motions of the material points, emerging as a result of the interatomic interaction of the rest of the body, is neglected. Moreover, the isolation of an arbitrary small part of the body to represent the whole clearly ignores the effects of the action of the applied load at a distance. These applied loads are important because their transmissions from one part of the body to another, through their common boundaries affect the motions, hence the state of the body at every point. The relevance of long-range or nonlocal outcomes on the mechanical properties of materials are well known. The main ideas of Nonlocal Continuum Mechanics were introduced during the 1960s (see [1] and references therein) and are based on considering the stress to be a function of the mean of the strain from a certain representative volume of the material centered at that point. Since then, there have been many situations of common occurrence wherein nonlocal effects seem to dominate the macroscopic behavior of matter. Interesting problems coming from a wide variety of areas such as damage and cracking analysis of materials, surface phenomena between two liquids or two phases, mechanics of liquid crystals, blood flow, dynamics of colloidal suspensions seem to demand this type of nonlocal approach which has made this area very active concerning recent developments in material and fluid science and engineering.

The true equation of state that describes the properties of matter at densities higher than nuclear ($\approx 10^{14}\ gr/cm^3$) is essentially unknown due to our inability to verify the microphysics of nuclear matter at such high densities [2]-[6]. Currently, what is known in this active field comes from the experimental insight and extrapolations from the ultra high energy accelerators and cosmic physics (see [7] and references therein). Having this uncertainty in mind, it seems reasonable to explore what is allowed by the laws of physics, in particular considering spherical or axial symmetries within the framework of the theory of General Relativity. In the present paper, we shall consider a *Nonlocal Equation of State* (*NLES* from now on) in general relativistic static spheres. It follows a previous work [8] on collapsing configurations having a *NLES*. The static limit of the particular *NLES* considered in this reference can be written as

$$P_r(r) = \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2\rho(\bar{r})\ \mathrm{d}\bar{r}\ + \frac{\mathcal{C}}{2\pi r^3}\ ; \tag{1}$$

where $\mathcal{C}$ is an arbitrary integration constant. It is clear that in equation (1) a collective behavior on the physical variables $\rho(r)$ and $P_r(r)$ is present. The radial pressure $P_r(r)$ is not only a function of the energy density, $\rho(r)$, at that point but also its functional throughout the rest of the configuration. Any change in the radial pressure takes into account the effects

of the variations of the energy density within the entire volume.

Additional physical insight of the nonlocality for this particular equation of state can be gained by considering equation (1) re-written as

$$P_r(r) = \rho(r) - \frac{2}{3}\langle\rho\rangle_r + \frac{\mathcal{C}}{r^3}\,, \qquad \text{with} \quad \langle\rho\rangle_r = \frac{\int_0^r 4\pi\bar{r}^2\rho(\bar{r})\,\mathrm{d}\bar{r}}{\frac{4\pi}{3}r^3} = \frac{M(r)}{V(r)}\,. \tag{2}$$

Clearly the nonlocal term represents an average of the function $\rho(r)$ over the volume enclosed by the radius r. Moreover, equation (2) can be easily rearranged as

$$P_r(r) = \frac{1}{3}\rho(r) + \frac{2}{3}\,\left(\rho(r) - \langle\rho(r)\rangle\right)\, + \frac{\mathcal{C}}{r^3} = \frac{1}{3}\rho(r) + \frac{2}{3}\,\sigma_\rho + \frac{\mathcal{C}}{r^3}\,, \tag{3}$$

where we have used the concept of statistical standard deviation σ_ρ from the local value of energy density. Furthermore, we may write:

$$P_r(r) = \mathcal{P}(r) + 2\sigma_{\mathcal{P}(r)} + \frac{\mathcal{C}}{r^3} \quad \text{where} \quad \begin{cases} \mathcal{P}(r) = \frac{1}{3}\rho(r) \qquad \text{and} \\ \\ \sigma_{\mathcal{P}(r)} = \left(\frac{1}{3}\rho(r) - \frac{1}{3}\langle\rho\rangle_r\right) = \left(\mathcal{P}(r) - \bar{\mathcal{P}}(r)\right)\,. \end{cases} \tag{4}$$

Therefore, if at a particular point within the distribution the value of the density, $\rho(r)$, gets very close to its average $\langle\rho(r)\rangle$ the equation of state of the material becomes similar to the typical radiation dominated environment, $P_r(r) \approx \mathcal{P}(r) \equiv \frac{1}{3}\rho(r)$.

In reference [8], it is shown that under particular circumstances a general relativistic spherically symmetric anisotropic (nonequal radial and tangential pressures, i.e. $P_r \neq P_\perp$) distribution of matter could satisfy a *NLES*. Some of these dynamic bounded matter configurations having a *NLES* with constant gravitational potentials at the surface, admit a Conformal Killing Vector and fulfill the energy conditions for anisotropic imperfect fluids. Several analytical and numerical models for collapsing radiating anisotropic spheres in general relativity were also developed in that paper. Although the perfect pascalian fluid assumption (i.e. $P_r = P_\perp$) is supported by solid observational and theoretical grounds, an increasing amount of theoretical evidence strongly suggests that a variety of very interesting physical phenomena may take place giving rise to local anisotropy. In the Newtonian regime, the consequences of local anisotropy originated by anisotropic velocity distributions have been pointed out in the classical paper by J.H. Jeans [9]. In the context of General Relativity, it was early remarked by G. Lemaître [10] that local anisotropy can relax the upper limits imposed on the maximum value of the surface gravitational potential. Since the pioneering work of R. Bowers and E. Liang [11], the influence of local anisotropy in General Relativity has been extensively studied (see [12] and references therein). Recently, the role of local anisotropy in the equation of state modeling the interior of extremely magnetized neutron stars or Magnetars at supranuclear densities, has been attracting the attention of researchers

(see [13]-[17] and references therein). It has been shown that these intense magnetic fields could induce local anisotropy in pressures within the equation of state of the matter distribution. In fact, two of these references[14] and [16] consider the consequences on the instability of the matter configuration in the most extreme case of anisotropy in pressures, i.e. when there exists a vanishing pressure perpendicular to the magnetic field. In this case the vanishing of the pressure transverse to the magnetic field leads the configuration to a transverse quantum collapse of the configuration.

The present paper is focused on the plausibility of obtaining static solutions satisfying *NLES* and whether those solutions could represent a credible bounded anisotropic matter distributions in General Relativity. We shall show that it is possible to obtain static anisotropic spherically symmetric matter configurations starting from known density profiles and satisfying a *NLES.* We also present two static solutions for *NLES* configurations concerning the particular case of vanishing radial pressure ($P_\perp \neq 0$ and $P_r = 0$) and vanishing tangential stresses, i.e., $P_\perp = 0$ and $P_r \neq 0$.

The structure of the present work is the following. The next section contains an outline of the general conventions, notation used, the metric and the corresponding field equations. Section 2 is devoted to solve the Einstein Field Equations for a matter distribution assuming a nonlocal equation of state. The family of *NLES* is presented in Section 3. In Section 4 we study the consequences imposed by the junction and energy conditions on bounded matter distribution. The method and some fluid sphere models are considered in the Section 5. The two particular cases, i.e. $P_r = 0$, and $P_\perp = 0$, are presented in Section 6. Finally, in the last section, our concluding remarks and results are summarized.

2 The Einstein Field Equations

To explore the feasibility of nonlocal equations of state for bounded configurations in General Relativity, we shall consider a static spherically symmetric anisotropic distribution of matter with an energy-momentum represented by $\mathbf{T}^{\mu}_{\nu} = diag\ (\rho, -P_r, -P_\perp, -P_\perp)$, where, ρ is the energy density, P_r the radial pressure and $P_\perp$ the tangential pressure.

We adopt standard Schwarzschild coordinates (t, r, θ, ϕ) where the line element can be written as

$$\mathrm{d}s^2 = e^{2\nu(r)}\mathrm{d}t^2 - e^{2\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{5}$$

with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, the solid angle.

The resulting Einstein equations are:

$$8\pi\rho = \frac{1}{r^2} + \frac{e^{-2\lambda}}{r}\left[2\lambda' - \frac{1}{r}\right], \tag{6}$$

$$-8\pi P_r = \frac{1}{r^2} - \frac{e^{-2\lambda}}{r}\left[2\nu' + \frac{1}{r}\right] \qquad \text{and} \tag{7}$$

$$-8\pi P_\perp = e^{-2\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \nu'' + \nu'\lambda' - (\nu')^2\right], \tag{8}$$

where primes denote differentiation with respect to r.

Using equations (7) and (8), or equivalently the conservation law $\mathbf{T}^{\mu}_{\nu;\mu} = 0$, we obtain the hydrostatic equilibrium equation for anisotropic fluids

$$P_r' = -(\rho + P_r)\,\nu' + \frac{2}{r}(P_\perp - P_r) \;. \tag{9}$$

Equation (6) can be formally integrated to give

$$e^{-2\lambda} = 1 - 2\frac{m(r)}{r}, \tag{10}$$

where a *mass function* $m(r)$, has been defined by

$$m(r) = 4\pi \int_0^r \rho\, \bar{r}^2\, \mathrm{d}\bar{r}\,, \tag{11}$$

and it corresponds to the mass inside a sphere of radius r as seen by a distant observer.

Finally, from (9), (10) and (7) the anisotropic Tolman-Oppenheimer-Volkov (TOV) equation [11] can be written as

$$\frac{\mathrm{d}\,P_r}{\mathrm{d}\,r} = -(\rho + P_r)\left(\frac{m + 4\pi r^3 P_r}{r\,(r - 2m)}\right) + \frac{2}{r}(P_\perp - P_r) \;. \tag{12}$$

Obviously, in the isotropic case ($P_\perp = P_r$) it becomes the usual TOV equation.

It has been established [18]-[20] that if ρ is a continuous positive function, $P_\perp(r)$ is a continuous differentiable function and $P_r(r)$ is a solution to the equation with starting value $P_\perp(0) = P_r(0)$, there exists a unique global solution to (12) representing a spherically symmetric fluid ball in General Relativity.

3 A Family of Solutions with a *NLES*

In this section we are going to present a family of static solution of the Einstein equations satisfying a *NLES*. Defining the new variables:

$$e^{2\nu(r)} = h\,(r)\; e^{4\beta(r)}, \quad \text{and} \qquad e^{2\lambda(r)} = \frac{1}{h\,(r)}; \qquad\qquad \text{with} \qquad h(r) \equiv 1 - 2\frac{m\,(r)}{r}\;, \tag{13}$$

the above metric (5) can be re-written as

$$\mathrm{d}s^2 = h\,(r)\; e^{4\beta(r)}\mathrm{d}t^2 - \frac{1}{h(r)}dr^2 - r^2\mathrm{d}\Omega^2\,, \tag{14}$$

The resulting Einstein Equations are:

$$8\pi\rho = \frac{1 - h - h'r}{r^2}\,, \tag{15}$$

$$8\pi P_r = -\frac{1 - h - h'r}{r^2} + \frac{4\,h\,\beta'}{r} \qquad \text{and} \tag{16}$$

$$8\pi P_\perp = \frac{h' + 2\,h\,\beta'}{r} + \frac{1}{2}\left[h'' + 4\,h\,\beta'' + 6\,h'\beta' + 8\,h\,\left(\beta'\right)^2\right]\,. \tag{17}$$

Now, equation (1) can re-stated as a differential equation

$$P_r(r) = \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2\rho(\bar{r})\;\mathrm{d}\bar{r}\; + \frac{\mathcal{C}}{2\pi r^3} \qquad \Leftrightarrow \qquad \rho - 3P_r + r\left(\rho' - P_r'\right) = 0\;. \tag{18}$$

Thus, from (18), (15) and (16), we have

$$\frac{2}{r}\left(h' + 2h\beta'\right) + h'' + 2\beta' h' + 2h\beta'' = 0\;, \tag{19}$$

which can be formally integrated yielding

$$\beta(r) = \frac{1}{2}\ln\left(\frac{\mathrm{C}}{h}\right) + \int \frac{\mathcal{C}}{r^2\,h}\;\mathrm{d}\,r + C_1\,, \tag{20}$$

where C and C_1 are arbitrary integration constants.

At this point, equation (20) deserves several comments:

- Firstly, if we set $\mathcal{C} = C_1 = 0$, a particular family of solutions ,

$$\beta(r) = \frac{1}{2}\ln\left(\frac{\mathrm{C}}{h}\right), \tag{21}$$

considered in a previous work [8] is found. The metric obtained in this particular static case recalls the so called isothermal coordinates system [21] which in turn is a particular case of the more general "warped space-time" (we refer the reader to [22] and references cited therein for a general discussion)

- Next, is the approach we have followed in order to obtain static anisotropic solutions having a *NLES*. It is clear that if the profile of the energy density, $\rho(r)$, is provided, the metric elements $h(r)$ and $\beta(r)$ can be calculated through (11), (13) and (20). Therefore, we can develop a consistent method to obtain static solutions having *NLES* from a known static ones.

- Finally, the metric elements (13) and (20), should fulfill the junction conditions and the physical variables coming from the energy momentum tensor, are only restricted by the hydrostatic equilibrium equation (12) and by some elementary criteria of physical acceptability. The next section is devoted to list these criteria for anisotropic and isotropic fluids with a *NLES*.

In terms of the metric elements (13) and 20, the corresponding Einstein Field Equations for anisotropic fluids having *NLES* can be written as:

$$8\pi\rho = \frac{2m'}{r^2}, \tag{22}$$

$$8\pi P_r = \frac{2m'}{r^2} - \frac{4\left(m - \mathcal{C}\right)}{r^3}, \qquad \text{and} \tag{23}$$

$$8\pi P_\perp = \frac{m''}{r} + \frac{2\left(m'r - m\right)}{r^3}\left[\frac{m'r - m}{r - 2m} - 1\right]. \tag{24}$$

4 Junction and Energy Conditions

Most exact solutions of the differential equation (12) supplied by the literature have been obtained from excessively simplifying assumptions solely with the purpose to find such solutions, and, consequently, they rarely represent physically "realistic" fluids (see for example, two interesting and complementary reviews on this subject: [23] and [24]). In order to establish the physical acceptability of the proposed static family of solutions (13) and (20), satisfying a *NLES,* (1) we state the consequences imposed by the junction and energy conditions for anisotropic fluids on bounded matter distribution.

We are going to consider bounded configurations, i.e. a matter distribution isolated in the sense that, where the pressure vanishes occurs at a finite radius. In fact, it can easily

be shown that the necessary and sufficient condition for matching an interior solution (14) onto the exterior Schwarzschild solution of total mass M,

$$\mathrm{d}s^2 = \left(1 - 2\frac{M}{r}\right)\mathrm{d}t^2 - \left(1 - 2\frac{M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{25}$$

is that the pressure equals zero at a finite radius $r = a$. In this case, the interior metric (14) should satisfy the following conditions at the boundary surface of the sphere $r = a$:

$$P_r(a) = 0 \qquad \Rightarrow \qquad \beta\,(a) = \beta_a = 0, \qquad \text{and} \qquad m(a) = M\ . \tag{26}$$

¿From the condition $P_r(a) = 0$, it is clear that

$$M = 4\pi\int_0^a \rho\,\bar{r}^2\,\mathrm{d}\bar{r}\ = 2\pi a^3\rho\,(a)\ . \tag{27}$$

Now, equation (23) leads to $\mathcal{C} = 0$ i.e.

$$m(r) = \mathcal{C} + 2\pi\left[\rho(r) - P_r(r)\right]r^3 \qquad \text{if} \quad m(0) = 0 \qquad \Rightarrow \quad \mathcal{C} = 0\ . \tag{28}$$

We should require that the metric elements have to be finite and non zero everywhere within the matter configuration, with no changes of sign nor loss of reality allowed. This implies that

$$m(r) > \frac{r}{2} \qquad\qquad \forall\ r\,, \tag{29}$$

where $m(r)$ is the mass function defined by equation (11). Recently [20] the above condition (29), for spherically static bounded configurations, has been proved to be equivalent to requiring that $\rho\ +\ P_r\ +\ 2\ P_\perp\ \geq\ 0$, (i.e. the *strong energy condition*) which clearly implies that the space time cannot contain a black hole region.

Next, we will establish the consequences of the physical acceptability for an anisotropic fluid configuration in terms of the mass function $m(r)$.

1. The density must be positive definite and its gradient must be negative everywhere within the matter distribution. Trivially, from equation (22) and its derivative we obtain

$$\rho > 0 \qquad \Rightarrow \qquad m' > 0 \qquad \text{and} \qquad \frac{\partial\rho}{\partial r} < 0 \qquad \Rightarrow \qquad m'' < \frac{2m'}{r}\ . \tag{30}$$

2. The radial and tangential pressure must be positive definite. Therefore, equations (23) and (24), yield

$$P_r \geq 0 \qquad \Rightarrow \qquad m' \geq \frac{2m}{r} \qquad \text{and} \tag{31}$$

$$P_\perp \geq 0 \qquad \Rightarrow \qquad m'' \geq -\frac{2\,(m'r - m)}{r^2}\left[\frac{m'r - m}{r - 2m} - 1\right]\ . \tag{32}$$

3. The radial pressure gradient must be negative definite. Thus, differentiating equation (23) we get

$$\frac{\partial P_r}{\partial r} \leq 0 \qquad \Rightarrow \qquad m'' \leq \frac{4m'}{r} - \frac{6m}{r^2}\ . \tag{33}$$

4. The speed of sound should be subluminal, consequently from equation (18) we obtain

$$v_s{}^2 \equiv \frac{\partial P_r}{\partial \rho} \leq 1 \qquad \Rightarrow \qquad \frac{(\rho - 3\,P_r\,)}{r\left(\frac{\partial \rho}{\partial r}\right)} \leq 0\ . \tag{34}$$

   It is clear that, due to the density gradient being negative everywhere within the configuration, the requirement of subluminal sound speeds leads to $\rho > 3\,P_r$ for both anisotropic and isotropic fluids having a *NLES*.

In addition to the above intuitive conditions we should satisfy either the *Strong Energy Condition* or *Dominant Energy Condition:*

- Demanding that the trace of the energy-momentum tensor be positive we find for the *Strong Energy Condition*, $\rho + P_r + 2\,P_\perp \geq 0$. Thus, using equations (22), (23) and (24) we get

$$\rho + P_r + 2\,P_\perp \geq 0 \qquad \Rightarrow m'' \geq -\frac{2m'}{r} + \frac{2m}{r^2} - \frac{2\,(m'r - m)}{r^2}\left[\frac{m'r - m}{r - 2m} - 1\right]. \tag{35}$$

  The Strong Energy Condition also implies $\rho + P_r \geq 0$ and $\rho + P_\perp \geq 0$, therefore,

$$\rho + P_r \geq 0 \qquad \Rightarrow \qquad m' \geq \frac{m}{r} \qquad \text{and} \tag{36}$$

$$\rho + P_\perp \geq 0 \qquad \Rightarrow \qquad m'' \geq -\frac{2m'}{r} - \frac{2\,(m'r - m)}{r^2}\left[\frac{m'r - m}{r - 2m} - 1\right]\ . \tag{37}$$

  In the case of isotropic fluids, the above condition (35) reads

$$\rho \geq 3\,P_r \qquad \Rightarrow \qquad m' \leq \frac{3m}{r}\,. \tag{38}$$

- The *Dominant Energy Condition* entails that the density must be larger than the pressure. Now, by using equations (22), (23) and (24) we get

$$\rho \geq P_r \qquad \Rightarrow \qquad m \geq 0 \qquad \text{and} \tag{39}$$

$$\rho \geq P_\perp \qquad \Rightarrow \qquad m'' \leq \frac{2m'}{r} - \frac{2\,(m'r - m)}{r^2}\left[\frac{m'r - m}{r - 2m} - 1\right]\ . \tag{40}$$

Notice that, if the density, the radial and the tangential pressures are positive definite and the Strong Energy Condition is satisfied, then the Dominant Energy Conditions (39) and (40) are automatically fulfilled, but the inverse is not true.

In order to determine the physical reasonableness of the anisotropic configurations having a *NLES* we shall explore two different sets of conditions. Both sets include conditions (29) through(34) but differ in the selection of the Strong Energy Condition or the Dominant Energy Condition. In the first group, we shall use the Strong Energy Condition (35), and because we require subluminal sound velocities (34), $\rho \geq 3\,P_r$ will be required. Obviously, when isotropic fluids are considered, the Strong Energy Condition reads $\rho \geq 3\,P_r$. and the sound speed will be subluminal in all cases. The second set again comprises conditions (29) through (34), the fulfillment of the Dominant Energy Conditions (equations (39) and (40)) and the requirement of $\rho \geq 3\,P_r$.

5 A Method for *NLES* Static Solutions

In the present section we shall state a general method to obtain *NLES* static anisotropic spherically symmetric solutions from known density profiles.

5.1 A Method for *NLES* Anisotropic Solutions

Concerning anisotropic fluids, i.e. non pascalian fluids where $P_r \neq P_\perp$, the method to obtain *NLES* static spherically symmetric solutions is as follows:

1. Select a static density profile $\rho(r)$, from a known static solution. Then, the mass distribution function, $m = m(r)$, can be obtained through equation (11). The junction condition (26) implies the continuity of $h(a)$ and the expression for the total mass $m(a) = M$ can be procured.

2. Next, check out where and under what circumstances, all the above physical and energy conditions, written in terms of the mass function, $m = m(r)$, are fulfilled. In other words, the mass distribution function obtained from the density profile selected should satisfy the inequalities (29) thought (36) (or (29) thought (34) and (39) and (40)), at least for some region $[r_1, r_2]$ within the matter distribution with $0 \leq r_1 \leq r_2 \leq a$, and for particular values of the physical parameters that characterize the configuration.

3. Following, the other metric coefficient, $\beta(r)$, can be found by using equation (20). Notice that because the boundary conditions (28) the actual expression for $\beta(r)$ will be (21).

4. Finally, Einstein Field Equations (23) through (24) provide the expressions for the radial and tangential pressures, P_r and $P_\perp$, respectively.

5. The integration constants are obtained as consequences of the junction conditions at the boundary surface, $r = a$, i.e. $\beta(a) = 0$, $m(a) = M$ and $P_r(a) = 0$.

In order to illustrate the above procedure we shall work out several examples for six static density profiles borrowed from the literature.

5.2 Examples of *NLES* Static Solutions

Example 1: The first example comes from a density profile proposed by B. W. Stewart [25], to describe anisotropic conformally flat static bounded configurations:

$$\rho = \frac{1}{8\,\pi r^2}\frac{\left(e^{2Kr}-1\right)\left(e^{4Kr}+8Kre^{2Kr}-1\right)}{\left(e^{2Kr}+1\right)^3} \qquad \Leftrightarrow \qquad m = \frac{r}{2}\left(\frac{e^{2Kr}-1}{e^{2Kr}+1}\right)^2 , \tag{41}$$

with $K = const.$

Thus, radial and tangential pressures can be written as

$$P_r = \frac{1}{8\pi r^2}\frac{\left(1-e^{2Kr}\right)\left(e^{4kr}-8Kre^{2Kr}-1\right)}{\left(1+e^{2Kr}\right)^3} \qquad \text{and} \qquad P_\perp = \frac{2K^2e^{4Kr}}{\pi\left[1+e^{2Kr}\right]^4} . \tag{42}$$

The constant K can be found from the boundary condition $M = m(r = a)$ as

$$K = \frac{1}{2a}\ln\left[\frac{1+\left(\frac{2M}{a}\right)^{\frac{1}{2}}}{1-\left(\frac{2M}{a}\right)^{\frac{1}{2}}}\right] , \tag{43}$$

while because the pressure vanishes at the surface $P_r(r = a) = 0$ we have:

$$e^{4Ka} - 8Kae^{2Ka} - 1 = 0 . \tag{44}$$

Example 2: The density profile of the second example is found due to P.S. Florides [26], but also, corresponding to different solutions, by Stewart [25] and more recently by M. K. Gokhroo and A. L. Mehra [27]. The Gokhroo-Mehra solution, represents densities and pressures which, under particular circumstances [28], give rise to an equation of state similar to the Bethe-Börner-Sato newtonian equation of state for nuclear matter [2, 3, 29].

$$\rho = \frac{\sigma}{8\pi}\left[1 - K\frac{r^2}{a^2}\right] \qquad \Leftrightarrow \qquad m = \frac{\sigma\,r^3}{6}\left[1\ - \frac{3\,K}{5}\frac{r^2}{a^2}\right] , \tag{45}$$

with σ and $K = const.$

The radial and tangential pressures are obtained from Einstein Field Equations (23) through (24) and can be expressed as

$$P_r = \frac{\sigma}{120\pi a^2}\left[5a^2 - 9Kr^2\right] \qquad \text{and} \tag{46}$$

$$P_\perp = \frac{\sigma}{120\pi a^2}\frac{18\sigma K^2 r^6 - 5a^2 r^2 K\left(3r^2+54\right) + 25a^4\left(\sigma r^2+3\right)}{5a^2\left(3-\sigma r^2\right)+3\sigma K r^4}\ , \tag{47}$$

where $\sigma = 0.631515$.

The boundary conditions at the surface $r = a$, i.e. $M = m(a)$ and $P_r(a) = 0$, lead to

$$K = \frac{5}{9} \qquad \text{and} \qquad M = \frac{\sigma a^3}{9}\ . \tag{48}$$

Example 3: This solution was discovered by H.B. Buchdahl [30] and rediscovered later by [31]. The corresponding density profile can be written as.

$$\rho = \frac{3C}{16\pi}\frac{3+Cr^2}{\left(1+Cr^2\right)^2} \qquad \Leftrightarrow \qquad m = \frac{3C}{4}\frac{r^3}{1+Cr^2}\ , \tag{49}$$

with $C = const.$

The method applied to the above density profile leads to

$$P_r = \frac{3C}{16\pi}\frac{1-Cr^2}{\left(1+Cr^2\right)^2} \qquad \text{and} \qquad P_\perp = \frac{3C}{16\pi}\frac{2-Cr^2+3\,C^2r^4}{\left(2-Cr^2\right)\left(1+Cr^2\right)^3}\ , \tag{50}$$

where $C = \frac{1}{a^2}$ coming from $P_r(a) = 0$

Example 4 The matter distribution sketched in this example is borrowed from Tolman IV isotropic static solution which was originally presented by R.C. Tolman in 1939 [32]. Tolman IV static solution is, in some aspects, similar to the equation of state for a Fermi gas in cases of intermediate central densities. This same profile (and Tolman IV solution) is also found as a particular case of a more general family of solution in [33] and [34]

$$\rho = \frac{C}{8\,\pi}\left[\frac{1-3K-3Kx}{(1+2\,x)} + \frac{2\,(1+Kx)}{\left(1+2\,x\right)^2}\right] \qquad \Leftrightarrow \qquad m = -\frac{x^{\frac{3}{2}}}{2C^{\frac{1}{2}}}\frac{K\,(1+x)-1}{(1+2x)}\ . \tag{51}$$

Einstein Field Equations (23) - (24) provide the expressions for the radial and tangential pressures as

$$P_r = \frac{C}{8\,\pi}\frac{1-2x-K\left(1+x+2x^2\right)}{\left(1+2x\right)^2} \qquad \text{and} \tag{52}$$

$$P_\perp = \frac{C}{8\pi}\frac{\left(4x^4+6x^3+10x^2+7x-1\right)K^2x - \left(4x^4+24x^3+19x^2+4x+1\right)K - 6x^2-3x+1}{\left(1+2x\right)^3\left(1+x\right)\left(1+Kx\right)}\ , \tag{53}$$

where $x = C\,r^2$;the constants K and C are obtained from the boundary conditions $M \;=\; m(a)$ and $P_r(a) \;=\; 0$ respectively, i.e.

$$K = \frac{{x_1}^{\frac{3}{2}} - 2MC^{\frac{1}{2}}\,(1+2x_1)}{x_1^{\frac{3}{2}}\,(1+x_1)} \qquad \text{and} \qquad M = \frac{1}{C^{\frac{1}{2}}}\frac{x_1^{\frac{5}{2}}}{1+x_1+2x_1^2}\ , \tag{54}$$

and $x_1 = C\,a^2$

Example 5: The density profile of this example corresponds to a solution originally proposed M. Wyman [35]. Again, the same solution is found in [33, 34, 36, 37] and [38]

$$\rho = -\frac{C}{8\,\pi}\frac{K\,(3+5\,x)}{(1+3\,x)^{\frac{5}{3}}} \qquad \Leftrightarrow \qquad m = -\frac{1}{2C^{\frac{1}{2}}}\frac{Kx^{\frac{3}{2}}}{(1+3x)^{\frac{2}{3}}}\ , \tag{55}$$

with $x = C\,r^2$ and $K, C = const.$

The method applied to the density profile (55) leads to radial and tangential pressures

$$P_r = \frac{C}{8\,\pi}\frac{K\,(x-1)}{(1+3x)^{\frac{5}{3}}} \qquad \text{and} \tag{56}$$

$$P_\perp = \frac{CK}{8\pi}\frac{(1+3x)^{\frac{2}{3}}\,(x^2+4x-1) + Kx\,(3x^2+8x+1)}{(1+3x)^{\frac{8}{3}}\left[Kx + (1+3x)^{\frac{2}{3}}\right]}\ , \tag{57}$$

where

$$K = -2^{\frac{7}{3}}\frac{M}{a} \qquad \text{and} \qquad x_1 = C\,a^2\ . \tag{58}$$

The constant K has been obtained for boundary conditions and the pressure at the surface $(P_r(a) \;=\; 0)$ determines the next integration constant: $C := \frac{1}{a^2}$

Example 6: The following density profile was found by M.P. Korkina, in 1981, [33] and rediscovered a year later by M.C. Durgapal [34]. It can be written as

$$\rho = \frac{C}{8\,\pi\,(1+\,x)^2}\left[\frac{3}{2}\,(3+x) - \frac{3K\,(1+3\,x)}{(1+4\,x)^{\frac{3}{2}}}\right] \Leftrightarrow m = -\frac{x^{\frac{3}{2}}}{4C^{\frac{1}{2}}}\frac{2K-3\,(1+4x)^{\frac{1}{2}}}{(1+x)\,(1+4x)^{\frac{1}{2}}}; \tag{59}$$

with $x = C\,r^2$ and $K, C = const.$

Again, for the density profile (59) the radial and tangential pressures are found to be

$$P_r = \frac{C}{8\,\pi}\frac{3\,(1+4x)^{\frac{3}{2}}\,(1-x) - 2K\,(1-x-8x^2)}{(1+x)^2\,(1+4x)^{\frac{3}{2}}} \qquad \text{and} \tag{60}$$

$$P_\perp = -\frac{C}{16\pi}\frac{4K^2x\,\Lambda + 4K\,(1+4x)^{\frac{1}{2}}\,\Xi + 3x\,\Pi + 6}{(1+x)^3\,(1+4x)^{\frac{5}{2}}\left[(1+4x)^{\frac{1}{2}}\,(x-2) - 2Kx\right]}\ , \tag{61}$$

where

$$\begin{aligned} \Lambda &= -8x^4 + 34x^3 + 36x^2 + 13x + 1\ , \\ \Xi &= 8x^5 - 65x^4 - 16x^3 + 5x^2 + x - 1 \qquad \text{and} \\ \Pi &= 192x^4 + 80x^3 + 116x^2 + 87x + 23\ . \end{aligned}$$

Finally, the constants K and C can be obtained for boundary conditions:

$$K = -\frac{(1+4\,x_1)^{\frac{1}{2}}}{2\,x_1^{\frac{3}{2}}}\left[4MC^{\frac{1}{2}}\,(1+x_1) - 3\,x_1^{\frac{3}{2}}\right]; \qquad \text{and} \qquad M = \frac{3\,x_1^{\frac{5}{2}}}{C^{\frac{1}{2}}\,(8x_1^2 + x_1 - 1)}\ , \tag{62}$$

with $x_1 = C\,a^2$

5.3 Modeling Anisotropic Spheres with NLES

The parameters: mass, M,in terms of solar mass $M_\odot$, M/a gravitational potential at the surface, boundary redshift z_a, surface density ρ_a and central density ρ_c, that characterize these bounded configurations are summarized in the following table

Equation of State	M/a	$M\ (M_\odot)$	z_a	$\rho_a \times 10^{14}\,(gr.cm^{-3})$	$\rho_c\ \times 10^{15}\ (gr.cm^{-3})$
Example 1	0.32	2.15	0.6	6.80	1.91
Example 2	0.40	2.80	1.2	8.84	1.99
Example 3	0.38	2.54	1.0	8.04	2.41
Example 4	0.25	1.69	0.4	5.36	2.00
Example 5	0.38	2.54	1.0	8.04	3.04
Example 6	0.35	2.37	0.8	7.75	2.11

All these parameters have been tuned up in order to that the mass function satisfies the physical and energy conditions, i.e., the inequalities (29) thought (36) (or (29) thought (34) and (39) and (40)), for a typical compact objects of radius of $a = 10$ Km..

6 $P_r = 0$, and $P_\perp = 0$ static solutions:

In this section we shall show that it is possible to obtain credible anisotropic spherically matter configuration with a *NLES* for two particular cases: one where the radial pressure vanishes, i.e., $P_\perp \neq 0$ and $P_r = 0$ and other where the tangential pressures vanishes, i.e., $P_\perp = 0$ and $P_r \neq 0$.

6.1 $P_r = 0$ and $P_\perp \neq 0$

The study of matter configurations with vanishing radial stresses traces back to G. Lemaître [10], A. Einstein[39] and P.S. Florides[26] in the static case. Non static models have been considered in the past in references [12, 40, 41, 42] and recently in [44, 45, 46, 48] concerning their relation with naked singularities. More recently conformally flat models with vanishing radial pressures has been considered in [43]. We shall present here one of these static solution satisfying an *NLES.*

From equation (23) it is trivial to see that

$$\frac{2m'}{r^2} = \frac{4m}{r^3} \qquad \Rightarrow \qquad m\,(r) = Cr^2. \tag{63}$$

Thus, density and tangential pressure profiles can be calculated as

$$\rho = \frac{C}{2\pi r} \qquad \text{and} \qquad P_\perp = \frac{C^2}{4\pi(1-2Cr)}\ . \tag{64}$$

The constant C can be obtained from the boundary conditions, i.e.

$$m(a) = M \qquad \Rightarrow \qquad C = \frac{M}{a^2}\ . \tag{65}$$

6.2 $P_\perp = 0$ and $P_r \neq 0$

The second static solution with a *NLES* we shall consider is the case $P_\perp = 0$ and $P_r \neq 0$. The rationale behind this supposition is the considerable effort that has been directed in recent years to study the effects of intense magnetic fields ($B \gtrsim 10^{15}$ G) on highly compact astrophysical objects ([13]-[17] and references therein). In fact, some observations seems to confirm that newly born neutron stars with very large surface magnetic fields (magnetar) could represent either soft gamma repeaters or anomalous X Ray pulsars [15], or both. One of the most striking effects of these large magnetic fields on the equation of state for superdense matter distributions is that they induce local anisotropic pressures[14, 16, 17] and it is possible to obtain extreme cases were the pressure perpendicular to the magnetic field

could vanish. In our case we shall suppose that tangential stresses vanish in all directions. In this sense we consider our solution inspired by the magnetar models

It is clear that if $P_\perp = 0$ equation (24) leads to

$$\frac{m''}{r} + \frac{2\left(m'r - m\right)}{r^3}\left[\frac{m'r - m}{r - 2m} - 1\right] = 0 \ , \tag{66}$$

which can be integrated yielding

$$m = \frac{r}{2}\left[1 - e^{-2(C_1 r + C_2)}\right] \ . \tag{67}$$

The density and the pressure profiles emerge from equations (11) and (1), respectively as:

$$\rho = \frac{e^{-2(C_1 r + C_2)}}{8\pi r^2}\left[2rC_1 - 1 + e^{2(C_1 r + C_2)}\right] \qquad \text{and} \tag{68}$$

$$P_r = \frac{e^{-2(C_1 r + C_2)}}{8\pi r^2}\left[2rC_1 + 1 - e^{2(C_1 r + C_2)}\right] \ . \tag{69}$$

The constants C_1 and C_2 can be obtained for boundary conditions at $r = a$ as

$$m(a) = M \qquad \Rightarrow \qquad C_1 = \frac{M}{a\left(a - 2M\right)} \qquad \text{and} \tag{70}$$

$$P_r(a) = 0 \qquad \Rightarrow \qquad C_2 = \frac{-M}{a - 2M} + \frac{1}{2}\ln\left(\frac{a}{a - 2M}\right) \ . \tag{71}$$

It is worth mentioning that from equation (67), we have

$$m = 0 \qquad \Rightarrow \qquad \begin{cases} r = 0 \\ \\ r = \frac{a}{2}\left[2 + \left(\frac{a}{M} - 2\right)\ln\left(1 - 2\frac{M}{a}\right)\right] \end{cases} \tag{72}$$

Therefore. matter configurations with *NLES* having vanishing tangential pressures are only possible within a region of the sphere, i.e.

$$\frac{a}{2}\left[2 + \left(\frac{a}{M} - 2\right)\ln\left(1 - 2\frac{M}{a}\right)\right] \ < \ r \ < \ a, \tag{73}$$

because,within the range $0 \ < \ r \ < \ \frac{a}{2}\left[2 + \left(\frac{a}{M} - 2\right)\ln\left(1 - 2\frac{M}{a}\right)\right]$ the mass function becomes negative.

7 Concluding Remarks

We presented a method to obtain *NLES* static anisotropic spherically symmetric exact solutions starting from known density profiles. It is clear that, when such a density profile, $\rho(r)$, is provided the radial pressure, $P_r(r)$, can be obtained from the *NLES*(1) and the tangential pressure $P_\perp(r)$ can be solved algebraically from the anisotropic Tolman-Oppenheimer-Volkov (TOV) differential equation (12). Althought, we supposed to have found a more general family of solutions for the Einstein Equations, (20), than the one considered in [8] but, due to the boundary conditions (28), the actual family of solutions is indeed the previous (21) considered in reference (21).

It is also obtained credible anisotropic spherically matter configurations with a *NLES* for two particular cases: one where the radial pressure vanishes, i.e., $P_\perp \neq 0$ and $P_r = 0$ and other where the tangential pressures vanishes, i.e., $P_\perp = 0$ and $P_r \neq 0$.

We worked out in detail the junction and energy conditions for anisotropic (and isotropic) bounded matter distributions, i.e. (29) thought (40), establishing their consequences in terms of the mass function $m(r)$ for bounded configurations having *NLES.* In the early days of Relativistic Astrophysics, some of these results led to the first general theorems due to H. A. Buchdahl[30, 49] and H. Bondi[50, 51] concerning inequalities limiting the behavior of the mass function for compact objects.

Because the mass function $m(r)$, presented in equations (41) through (59) and the corresponding pressure profiles are regular at the origin, the solutions to the Einstein Equations are considered unique in the sense stated by Rendall and collaborators[18, 19]. That is to say: for a given value of the central pressure, $P_0 = P_\perp(r=0) = P_r(r=0)$, there exists a unique global solution $P_r(r)$ to the anisotropic TOV equation (12) representing a spherically symmetric fluid sphere in General Relativity.

It is clear that the last term in the anisotropic TOV equation (12), $\frac{2}{r}(P_\perp - P_r) \equiv \Delta$, represents a "force" due to the local anisotropy. This "force" is directed outward when $P_\perp > P_r \Leftrightarrow \Delta > 0$ and inward if $P_\perp < P_r \Leftrightarrow \Delta < 0$. As it is apparent from the figure and also from the above table, those models with a (average) repulsive "anisotropic" force ($\Delta > 0$) allows the construction of more massive distributions. This picture is more evident for solutions where $P_\perp \neq 0$ and $P_r = 0$ and $P_\perp = 0$ and $P_r \neq 0$, which are the extreme situations. Considering plausible static matter configurations (it is to say those that satisfy conditions (29) thought (40)) we have

- more massive configurations if the radial pressure vanishes i.e. $\Delta > 0$ has a maximum,
- less massive configurations if the tangential pressures vanishes i.e. $\Delta < 0$ has a minimum.

Notice that Model 4 (*NLES* Tolman IV anisotropic static solution) and Model 5 (*NLES* anisotropic Wyman like static solution) have "soft" ($P_\perp < P_r$) cores and "hard" ($P_\perp > P_r$)

outer mantles. It is worth mentioning that models presented here are very sensitive to the change of values of the parameters sketched in the table.

As it can be appreciated from equation (72), fluids satisfying *NLES* and having vanishing tangential pressures are only possible within a region of the matter configuration with $\frac{a}{2}\left[2+\left(\frac{a}{M}-2\right)\ln\left(1-2\frac{M}{a}\right)\right] \ < \ r \ < \ a$. Within the inner core, the mass function becomes negative. This situation emerges from a limitation of the definition of the Schwarzschild mass function (11) when it is apply to anisotropic fluid configurations. It is interesting to say that the expression of the Tolman Whittaker mass[52]

$$m_{TW}\left(r\right) = 4\pi\int_0^r r^2 e^{(\nu+\lambda)/2}\left(\rho - P_r - 2P_\perp\right)\mathrm{d}r \equiv e^{(\nu+\lambda)/2}\left(m + 4\pi P_r r^3\right) \tag{74}$$

for this static *NLES* model reads

$$m_{TW}\left(r\right) = C_1\ r^2 \tag{75}$$

and precludes non physical situations at the core of the distribution. It is clear that for this case the Tolman-Whittaker mass function, m_{TW}, is a more suitable concept of mass than the Schwarzschild mass function, m (see [12] for a comprehensive discussion on this point, plentiful of examples). More over, from equations (63) and (75), the expression of the m_{TW} for this particular static *NLES* model with vanishing tangential stresses is the same that m for the model with vanishing radial pressure.

It can also be mentioned as a curiosity that the relation among the areas under the curves displayed in the figure, i.e. $\int_0^R \frac{2\mathrm{d}r}{r}\left(P_\perp - P_r\right)$, corresponds to the relation among de total masses. The more massive the model is the greater the area under the curve $\frac{2}{r}\left(P_\perp - P_r\right)$ is.

In principle, a *NLES* in isotropic pascalian fluid ($P_\perp = P_r$) can not be ruled out, but we have integrated numerically the standard TOV equation (12), which can be written in terms of the mass distribution function as

$$m'' = \frac{2m'}{r} - \frac{4m}{r^2} - \frac{2\left(m'r - m\right)}{r^2}\left[\frac{m'r - m}{r - 2m} - 1\right] \qquad \text{with} \qquad \left\{\begin{array}{l} m(0) = 0 \\ \\ m'(0) = 0 \end{array}\right. \tag{76}$$

and the inequalities (29) thought (36) (or (29) thought (34) and (39) & (40)) corresponding the physical and energy conditions **were not** fulfilled for any configuration having a *NLES*.

8 Acknowledgments

We are indebted to J. Flórez López for pointing out us the relevance of nonlocal theories in modern classical continuum mechanics and to L. Herrera Cometta for the references of Magnetars. We also gratefully acknowledge the financial support of the Consejo de Desarrollo

Científico Humanístico y Tecnológico de la Universidad de Los Andes under project C-1009-00-05-A, and to the Consejo Nacional de Investigaciones Científicas y Tecnológicas under project S1-2000000820

References

[1] Narasimhan, M.N.L., (1993), *Principles of Continuum Mechanics*, (John Willey, New York) p. 510.

[2] Demiański M., (1985), *Relativistic Astrophysics,* in International Series in Natural Philosophy, Vol 110, Edited by *D. Ter Haar*, (Pergamon Press, Oxford).

[3] Shapiro S.L. and Teukolsky S.A., (1983), *Black Holes, White Dwarfs and Neutron Stars*, (John Willey, New York).

[4] Balberg, S. and Shapiro, S.L. (2000) *The Properties of Matter in White Dwarfs and Neutron Stars* in Handbook of Elastic Properties, Edited by *H.E. Bass, V.M. Keppens, M. Levy and R. Raspet (* Academic Press, New York*).* [On line Los Alamos Archive Preprint]. Cited on 24 Apr, 2000, `http://xxx.lanl.gov/abs/astro-ph/0004317.`

[5] Kippenhahn, R. and Weigert, A., (1990), *Stellar Structure and Evolution*, (Springer Verlag, New York).

[6] Glendenning, N. K. (2000), *Compact Stars* (Springer Verlag, New York)

[7] Heiselberg, H. and Pandharipande V. (2000) *Ann. Rev. Nucl. Part. Sci.* **50,** 481.[On line Los Alamos Archive Preprint]. Cited on 20 Mar, 2000, `http://xxx.lanl.gov/abs/astro-ph/0003276.`

[8] Hernández, H., Núñez, L.A., and Percoco, U. (1999), *Class. Quantum Grav*, **16**, 871. [On line Los Alamos Archive Preprint]. Cited on 5 June, 1998, `http://xxx.lanl.gov/abs/gr-qc/9806029.`

[9] Jeans, J.H. (1922), *Mon. Not. R Astron. Soc.*, **82**, 122.

[10] Lemaître, G. (1933), *Ann. Soc. Sci. Bruxelles,* **A53**, 51.

[11] Bower, R. and Liang, E. P. (1974), *Astrophys. J.*, **188,** 657.

[12] Herrera, L. and Santos, N.O., (1997), *Physics Reports,* **286**, 53.

[13] Suh, I-S.,and Mathews, G. J. (1999), Nuclear Equation of State and Internal Structure of Magnetars, in *Proceedings of the 5th Huntsville Gamma Ray Burst Symposium*, Huntsville, Alabama, USA, Oct. 18-22, 1999

[14] Chaichian, M., Masood, S.S.,Montonen, C., Perez-Martínez, A. and Pérez-Rojas, H. (2000), *Phys.Rev.Lett.*, **84**, 5261. [On line Los Alamos Archive Preprint]. Cited on 3 Nov 1999, `http://lanl.arXiv.org/abs/hep-ph/9911218`

[15] Suh, I-S., and Mathews, G. J. (2001), *Astrophys.J.* **546**, 1126. [On line Los Alamos Archive Preprint]. Cited on 1 Sep 2000, `http://lanl.arXiv.org/abs/astro-ph/9912301`

[16] Pérez Martínez, A., Pérez Rojas, H., and Mosquera, H., (2000), [On line Los Alamos Archive Preprint]. Cited on 30 Nov 2000, `http://lanl.arXiv.org/abs/hep-ph/0011399`

[17] Kohri, K., Yamada, S. and Nagataki, S. (2001), [On line Los Alamos Archive Preprint]. Cited on 25 Jun 2001, `http://xxx.lanl.gov/abs/hep-ph/0106271`

[18] Rendall, A.D. and Schmidt, B.G. (1991), *Class. Quantum Grav*, **8**, 985.

[19] Baumgarte, T.W. and Rendall A.D. (1993), *Class. Quantum Grav.*, **10**, 327.

[20] Mars, M., Mercè Martín-Prats, M. and Senovilla, J.M. (1996), *Phys. Lett. A*, **218,** 147. [On line Los Alamos Archive Preprint]. Cited on 1 Feb 2002, `http://lanl.arXiv.org/abs/gr-qc/0202003`.

[21] Synge, J.L. (1960), *Relativiy: The General Theory* (North Holland, Amsterdan), p 269.

[22] Carot, J. and da Costa, J. (1993), *Class. Quantum Grav.* **10**, 461 .

[23] Finch, M.R. and Skea, J.F.E. *A Review of the Relativistic Static Fluid Spheres* (preprint available on the World Wide Web at `http://edradour.symbcomp.uerj.br/pubs.html`)

[24] Delgaty, M.S.R. and Lake, K. (1998), *Comput. Phys. Commun.*, **115**, 395.[On line Los Alamos Archive Preprint]. Cited on 2 Sep 1998, `http://xxx.lanl.gov/abs/gr-qc/9809013.`

[25] Stewart, B.W. (1982), *J Phys. A. Math Gen.*, **15**, 2419.

[26] Florides, P.S. (1974), *Proc. Roy. Soc. Lond.*, **A337**, 529.

[27] Gokhroo, M.K. and Mehra, A.L. (1994), *Gen. Rel. Grav.*, **26**, 75.

[28] Martínez J., (1996), *Phys. Rev. D*, **53** , 6921.

[29] Bethe, H.A., Börner, G. and Sato, K. (1970) *Astr. and Ap* **7**, 279.

[30] Buchdahl, H.A., (1959), *Phys. Rev.* **116**, 1027.

[31] Durgapal, M.C. and Bannerji R (1983) *Phys. Rev. D*, **27**, 328.

[32] Tolman, R.C. (1939), *Phys. Rev.* **55**, 364.

[33] Korkina, M.P. (1981), *Sov. Phys J.* **24**, 468.

[34] Durgapal, M.C. (1982), *J Phys. A. Math Gen.*, **15**, 2637.

[35] Wyman, M., (1949), *Phys. Rev.* **75**, 1930.

[36] Kuchowicz, Br (1970), *Acta. Phys. Polon.,* **B1**, 437.

[37] Adler, R.J. (1974), *J. Math. Phys.* **15**, 727. (Erratum (1976), , *J. Math. Phys.* **17**, 158).

[38] Adams, R.C. and Cohen J.M. (1975), *Astrophys. J.*, **198,** 507.

[39] Einstein, A., (1939) *Ann. Math.* **40**, 4, 922.

[40] Datta, B.K., (1970) *Gen. Rel. Grav.* **1**, p.19.

[41] Bondi, H., (1971) *Gen. Rel. Grav.* **2**, 321.

[42] Evans A. (1977) *Gen. Rel. Grav.* **8**,155.

[43] Herrera, L., Di Prisco, A., Ospino J., and Fuenmayor, E. (2001), *J.Math.Phys.* **42**, 2129. [On line Los Alamos Archive Preprint]. Cited on 13 Feb 2001, `http://xxx.lanl.gov/abs/gr-qc/0102058`

[44] Magli, G. (1997), *Class. Quantum Grav.* **14**, 1937; **15**, 3215; Magli G. (1998) *Class.Quant.Grav.* **15** 3215. [On line Los Alamos Archive Preprint]. Cited on 18 Dec 1998, `http://xxx.lanl.gov/abs/gr-qc/9711082`

[45] Harada, T., Nakao, K., and Iguchi, H. (1999) *Class.Quant.Grav.* **16**, 2785 [On line Los Alamos Archive Preprint]. Cited on Thu, 20 May 1999, `http://lanl.arXiv.org/abs/gr-qc/9904073`

[46] Jhingan, S. and Magli, G. (2000) *Phys. Rev.D* **61**, 124006; [On line Los Alamos Archive Preprint]. Cited on 10 Jan 2000, `http://xxx.lanl.gov/abs/gr-qc/9902041`

[47] Jhingan, S. and Magli, G. (1999) [On line Los Alamos Archive Preprint]. Cited on 26 May 1999, `http://xxx.lanl.gov/abs/gr-qc/9903103`

[48] Barve, S., Singh, T. and Witten, L. (2000), *Gen. Rel. Grav.* **32**,697. [On line Los Alamos Archive Preprint]. Cited on 28 Jan 1999, `http://lanl.arXiv.org/abs/gr-qc/9901080`

[49] Buchdahl, H.A., (1981), *Seventeen Simple Lectures on General Relativity* (John Willey, New York).

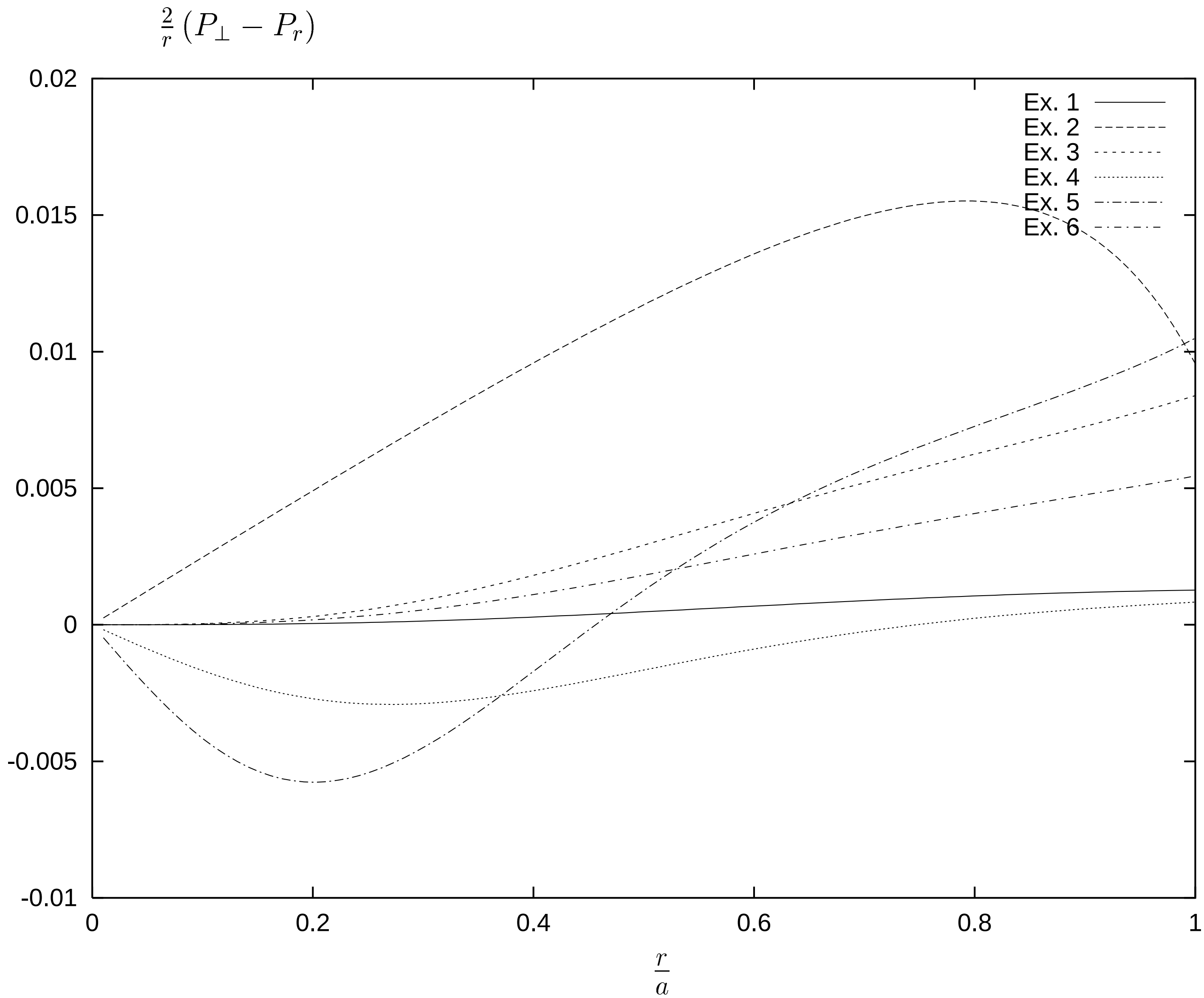
[50] Bondi, H. (1964) *Proc. Roy. Soc. (London)* **284**, 303.

[51] Trautman, A., Pirani, F.A.E. and Bondi H. 1965 *Lectures on General Relativity,* Brandeis 1964 Summer Institute on Theoretical Physics vol 1 (Prentice Hall Englewood Cliffs, N.J.)

[52] Tolman, R.C. (1930), *Phys. Rev.* **35**, 875; Whittaker, E. T. (1935), *Proc. R. Soc. London* **A149**, 384.

Figure caption

Figure 1: $\frac{2}{r}\left(P_\perp - P_r\right) \equiv \Delta$, as function of r/a which represents a "force" due to the local anisotropy. This "force" is directed outward when $P_\perp > P_r \Leftrightarrow \Delta > 0$ and inward if $P_\perp < P_r \Leftrightarrow \Delta < 0$. Model 4 (*NLES* Tolman IV anisotropic static solution) and Model 5 (*NLES* anisotropic Wyman like static solution) have "soft" ($P_\perp < P_r$) cores and "hard" ($P_\perp > P_r$) outer mantles.



$\frac{2}{r}\left(P_\perp - P_r\right)$
0.02
0.015
0.01
0.005
0
-0.005
-0.01
0
0.2
0.4
0.6
0.8
1
$\frac{r}{a}$
Ex. 1
Ex. 2
Ex. 3
Ex. 4
Ex. 5
Ex. 6